Computing Temporary Equilibria using Exact Aggregation*

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Abstract

We suggest a new method of approximating temporary equilibria in heterogeneous agent models. Our approach offers a significant speedup without a notable drop in accuracy relative to established methods. We demonstrate the effectiveness of our procedure by applying it to a model with heterogeneous boundedly rational agents, and comparing its performance to that of alternative methods.

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1 Introduction

Heterogeneous agent models based on Huggett (1993), Aiyagari (1994), and Krusell et al. (1998) have become standard in macroeconomics. A major challenge with such models is their reliance on computational methods for finding equilibrium solutions and simulating recursive dynamic equilibria, which can be time-consuming and costly (Algan et al., 2014).

When simulating heterogeneous agent (HA) economies, one must solve for the temporary equilibrium (TE) given the realization of state variables at each period of the simulation. In a TE, agents' decisions and prices are simultaneously pinned down by market clearing conditions – in turn, these outcomes determine aggregate dynamics. Practically, this involves finding agent-level policy rules as a function of individual and aggregate states, along with prices, then solving for equilibrium prices by applying a sequential root solver to a set of market clearing conditions that depend on the aggregate of individual policy rules as a function of aggregate states and prices (Bakota, 2022). As described in Den Haan et al. (2010), the repeated aggregation of individual policy rules during the computation of TE in a simulation may take a long time, even while using standard projection methods outlined in Judd (1992) and Judd (1996) to approximate individual policy functions.

We suggest a fast new approach to approximating TE in HA models that builds on existing projection methods. The key is to store the components of the aggregation of individual policy rule approximations that are independent of prices and aggregate states, such that each iteration of the root solver performs a minimal number of operations to compute aggregate outcomes. We find that our approach significantly outperforms existing aggregation methods, especially in cases with high-dimensional idiosyncratic state spaces, while avoiding notable losses in accuracy.

In Section 2, we build up to a formulation of our aggregation method by (1) generalizing the TE of an HA model; (2) mapping a specific example of a high-dimensional boundedly rational HA model to the general framework; and (3) describing two standard computational methods for approximating TE, and proposing a faster approach. In Section 3, we apply all three methods to the example model and compare their execution times and accuracy. The final section concludes the paper.

2 Methodology

Suppose we want to find the TE of an HA economy with idiosyncratic and aggregate states at some given point in time. We have a set of demand functions $\widetilde{x}(z,P;Z)$, which depend on individual states z with distribution Ω, prices P, and an aggregate state Z. The aggregate state follows a given law of motion $Z' = G(Z,E)$, where Z' denotes the realization of the aggregate state in the next period, and E denotes a vector of aggregate shocks. Our objective is to find the set of market-clearing prices P^* given Z, such that

$$\widetilde{X}(P^*;Z) \equiv \int \tilde{x}(z,P^*;Z)\, d\Omega(z) = \overline{X}(P^*;Z)\,, \tag{1}$$

where $\widetilde{X}$ and $\overline{X}$ represent aggregate demand and supply, respectively.s Next, we put this general framework in perspective by constructing a specific example of a Krussell-Smith model with endogenous labor supply and boundedly rational agents, as presented in Evans et al. (2023), in which agents have heterogeneous beliefs in addition to idiosyncratic productivity. We use this model as a benchmark to compare the execution times of competing TE approximation methodologies.

2.1 Model

Let time be discrete. The economy is populated by a continuum of agents, such that a given agent is endowed with a unit of labor per period and derives utility from consumption c and leisure l according to the instantaneous utility function $u(c,l)$. Each agent has a unique effective unit of labor for each unit of nominal labor supplied, and receives a corresponding wage that can be separated into the following two components: (1) a common aggregate component w; and (2) an idiosyncratic efficiency component ε that is i.i.d. across the population. We assume $\{\varepsilon\}$ to be a Markov process with time-invariant transition function Π. In each period, an agent can trade one-period claims to capital for net return r, limited by the exogenous borrowing constraint $\underline{a}$. Goods and factor markets are assumed to be competitive.

In period t, an agent holds claims a, experiences idiosyncratic efficiency ε, and faces factor prices r_t and w_t. Additionally, an agent has a vector of beliefs, $\psi \in \mathbb{R}^n$, which comprise of the coefficients of the forecasting model used to form expectations of next period's shadow price λ_{t+1}, where

$$\lambda_t(a,\varepsilon,\psi) \equiv (1+r_t)\, u_c(c_t(a,\varepsilon,\psi), l_t(a,\varepsilon,\psi)) \,. \tag{2}$$

All agents observe some common vector of aggregates $X_t \in \mathbb{R}^n$, and condition their individual forecasts of λ_{t+1} at time t, λ_t^e, on these aggregates. Each agent forms a forecast using the following perceived law of motion (PLM):

$$\log \lambda_t^e = \log \overline{\lambda}^e + \langle \psi, X_{t-1} \rangle \,, \tag{3}$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^n$, and $\overline{\lambda}^e$ is a time-t forecast of λ_{t+1} in a corresponding stationary recursive equilibrium without any aggregate risk. We may express

$$\overline{\lambda}^e(a', \varepsilon) = \int \overline{\lambda}(a', \varepsilon') \Pi(\varepsilon, d\varepsilon') \,, \tag{4}$$

where $\overline{\lambda}(a,\varepsilon) = (1+\bar{r}) u_c(\bar{c}(a,\varepsilon), \bar{l}(a,\varepsilon))$, such that $\bar{c}$, $\bar{l}$, and $\bar{r}$ represent the stationary equilibrium levels of consumption, labor, and the capital rate of return, respectively.[1]

Given factor prices r_t and w_t, each agent uses their forecast rule to determine their period-t decisions – $c_t(a,\varepsilon,\psi)$, $l_t(a,\varepsilon,\psi)$, and $a_t(a,\varepsilon,\psi)$ – which satisfy the following conditions:

$$u_c(c_t(a,\varepsilon,\psi), l_t(a,\varepsilon,\psi)) \geq \beta\, \lambda_t^e(a_t(a,\varepsilon,\psi), \varepsilon, \psi) \tag{5}$$
$$\text{and } a_t(a,\varepsilon,\psi) \geq \underline{a}, \text{ with c.s.}$$
$$u_l(c_t(a,\varepsilon,\psi), l_t(a,\varepsilon,\psi)) = u_c(c_t(a,\varepsilon,\psi), l_t(a,\varepsilon,\psi)) w_t \tag{6}$$
$$a_t(a,\varepsilon,\psi) = (1+r_t)a + w_t \cdot \varepsilon \cdot (1 - l_t(a,\varepsilon,\psi)) - c_t(a,\varepsilon,\psi) \,. \tag{7}$$

The representative firm rents capital k_t at real rental rate $r_t + \delta$, hires effective labor

[1] Essentially, we assume that agents' forecasts of the following period's shadow price consists of two components: (1) a rational forecast of next period's shadow price in a corresponding stationary equilibrium without aggregate risk; and (2) a boundedly rational forecast of the expected deviation of next period's shadow price from its stationary level attributed to variation in aggregate risk, as predicted by variation in the observed aggregate variables. For an in-depth discussion of this PLM and how agents update their beliefs, refer to Evans and McGough (2021) and Evans et al. (2023). For the purposes of this paper – to approximate a TE – we treat the distribution of ψ over the population of agents as given.

n_t at real wage w_t, and produces output under perfect competition using CRTS technology $\theta f(k,n)$, where δ is the capital depreciation rate. We take $\{\theta_t\}$ to be a stationary process that affects total factor productivity, with dynamics given by $\theta_{t+1} = \nu_t \theta_t^{\rho}$, $|\rho| < 1$, and $\{\nu_t\}$ is iid having log-normal distribution. There are no capital installation costs. Profit maximization behavior by the firm implies that factors earn their marginal products:

$$w_t = \theta_t f_n(k_t, n_t) \quad \text{and} \quad r_t + \delta = \theta_t f_k(k_t, n_t). \tag{8}$$

Given agent-specific states and beliefs (a, ε, ψ), and observable aggregates X_t, the conditions (5)–(7) determine agents' decision schedules in terms of prices (r_t, w_t). The realized values of prices and other endogenous aggregates are determined by market clearing, i.e. TE. Mechanically, this determination requires tracking the evolving distribution of agent-specific states *and* agent-specific beliefs. Let μ_t be the contemporaneous distribution of agent-states and beliefs. Then TE imposes that $r_t = \theta_t f_k(k_t, n_t) - \delta$ and $w_t = \theta_t f_n(k_t, n_t)$, where k_t and n_t are determined by the market clearing conditions

$$k_t = \int a \cdot \mu_t(da, d\varepsilon, d\psi) \quad \text{and} \quad n_t = \int (1 - l_t(a, \varepsilon, \psi))\, \mu_t(da, d\varepsilon, d\psi), \tag{9}$$

and θ_t is the realized TFP shock. The n_t in the above equation depends on the policy rules $l_t(a, \varepsilon, \psi)$, which, in turn, depend implicitly on current factor prices (r_t, w_t). All must be jointly determined in the TE as solutions to a system of non-linear equations.

We now map this particular model to the general class of TE described at the beginning of this Section. For a given agent, the set of individual states is $z = (a, \varepsilon, \psi)$ with some distribution μ_t, the set of prices faced by all agents is $P = (r_t, w_t)$, and the aggregate state is $Z = (\mu_t, \theta_t, X_{t-1})$ with transition dynamics governed by $Z' = H(Z, \nu_{t+1})$. Therefore, an agent's set of demand functions is captured by $\tilde{x}(z,P,Z) = (c_t(z,P,Z), l_t(z,P,Z), a_t(z,P,Z))$. We may therefore express the market clearing conditions presented in Eq. (1) by

$$k_t = \int a \cdot \mu_t(dz) \quad \text{and} \quad n_t = \int (1 - l_t(z,P,Z))\, \mu_t(dz). \tag{10}$$

In our application, beliefs ψ are fixed, hence the aggregate observables X_{t-1} have no impact on the equilibrium outcome. Furthermore, we fix the TFP shock with $\theta_t = 1$, so that it can be disregarded. Finally, we initialize the distribution μ_t of individual states $z = (a, \varepsilon, \psi)$ by fixing it to some $\widehat{\mu}$.

2.2 Approximation Methods

We present three distinct approaches to approximating TE in HA models, and apply all three to our example model. We find that our novel method significantly outperforms the rest in terms of execution time, without suffering from a notable loss in accuracy.

Method #1 (Naive Global Approximation): This is the most direct of all methods. Let Ω be approximated with N points $\{\bar{z}_i\}_{i \in \mathbb{N}_N}$ with corresponding weights $\{\omega_i\}_{i \in \mathbb{N}_N}$. The approximation for the demand schedule $\widetilde{X}(P;Z) \in \mathbb{R}^J$ for a given P and Z may be expressed as the following weighted sum:

$$\widehat{X}^j(P;Z) \equiv \sum_i \omega_i \tilde{x}^j(\bar{z}_i, P; Z), \tag{11}$$

where $\tilde{x}$ is derived individually for each $(\bar{z}_i, P; Z)$ tuple, and $j = 1, \ldots, J$ is an index for the set of goods in the economy. Repeating the above approximation for all J goods yields $\widetilde{X}(P;Z)$, which can then be used to solve for the equilibrium vector of prices P^* using the market clearing condition in Eq. (1).

The benefit of this approach is its precision – all of the policy functions are solved directly, therefore $\widehat{X}$ and $\widetilde{X}$ are likely to be close. On the other hand, solving the policy functions N times for a given P can be computationally taxing, especially if N is large. If an M number of steps is required for a root solver to converge to an approximation of P^*, then each $\tilde{x}^j$ needs to be recomputed a total of $M \cdot N$ number of times.

In our application, for a given wage level w and distribution $\widehat{\mu}$ of individual states (a, ε, ψ), we analytically solve for the level of capital k and return on capital r. Using the above objects, we then solve for the quantity of labor supplied individually by each agent, $1 - l_i$, according to conditions (5)–(7). Averaging these individual labor supply decisions across the set of all agents, as in Eq. (11), yields the aggregate labor supply n_t, as shown in Eq (9).

Method #2 (Interpolated Global Approximation): This method adds another layer of approximation to the first. Suppose that $\tilde{x}$ is approximated by $\hat{x}$ via projection, such that the demand for the j-th good is represented by

$$\hat{x}^j(z,P;Z) = \sum_k \sum_l \sum_m c^j_{kl}\, \Phi^z_k(z)\, \Phi^P_l(P)\, \Phi^Z_m(Z)\,, \tag{12}$$

where Φ^P_l, Φ^z_k and Φ^Z_m are the basis functions that each depend on prices, idiosyncratic states, and the aggregate state, respectively. The approximation for the demand schedule $\widetilde{X}(P;Z)$ may be expressed as the following weighted sum:

$$\widehat{X}^j(P;Z) \equiv \sum_i \omega_i \hat{x}^j(\bar{z}_i,P)\,, \tag{13}$$

where $\bar{z}_i$ and ω_i are defined as before. Once again, Eq. (13) may be used to solve for P^* in Eq. (1).

The attractiveness of this approach lies in that it requires the policy function for the j-th good to be approximated only once, after which it is inputted into Eq. (13) to compute the sum. However, given a large N, the sum in Eq. (13) may still take a substantial amount of time to compute. This is especially true if the vector of idiosyncratic states z is high-dimensional, in which case each instance of a pre-computed $\hat{x}$ may take a long time to execute – causing Method 1 to outpace Method 2.[2] Furthermore, this method likely provides a less accurate solution for $\widetilde{X}$ relative to Method 1, due to the additional layer of approximation.

In applying this method to our example model, we essentially repeat the same steps as with Method 1. However, we first approximate the labor supply policy function of each agent i by interpolating them over grids of the idiosyncratic states (a, ε, ψ) and prices (r, w) centered around their stationary recursive equilibrium values. After solving for k and r, as before, we solve for the quantity of labor supplied individually by each agent – this time by inputting their corresponding idiosyncratic states. Averaging these individual labor supply decisions across the set of all agents gives us the aggregate labor supply n_t.

Method #3 (Fast Approximation): Finally, we present our new method of approximating TE, which is based on an algebraic manipulation of Method 2. Notice that substituting Eq.

[2] We demonstrate this in our application.

(12) into Eq. (13) yields

$$\widehat{X}^j(P;Z) = \sum_i \omega_i \sum_l \sum_k \sum_m c^j_{kl}\, \Phi^z_k(z)\, \Phi^P_l(P)\, \Phi^Z_m(Z)\,, \tag{14}$$

which can be rearranged by distributing ω_i, switching the order of summation, and factoring Φ^P_l in the following manner:

$$\widehat{X}^j(P;Z) = \sum_l \Phi^P_l(P) \sum_i \omega_i \sum_k \sum_m c^j_{kl} \Phi^z_k(z) \Phi^Z_m(Z)\,. \tag{15}$$

Finally, letting $C^j_l(Z) \equiv \sum_i \omega_i \sum_k \sum_m c^j_{kl}\, \Phi^z_k(z)\, \Phi^Z_m(Z)$ allows us to express Eq. (15) as

$$\widehat{X}^j(P;Z) = \sum_l C^j_l(Z)\, \Phi^P_l(P)\,. \tag{16}$$

With this formulation of $\widehat{X}$, since C^j_l is independent of P, it is sufficient to compute C^j_l once before initializing the root solver to find P^*. This approach strictly dominates Method 2 by relying on the same projections while being significantly faster – it also likely outpaces Method 1 even when z is high-dimensional, as is shown in Section 3.

It is worth mentioning that the execution speed of a TE solver becomes crucial in practice when the dynamic recursive equilibrium of a heterogeneous agent economy is being simulated over some period of time – in other words, when the TE solver is used repeatedly (Bakota, 2022). For such applications, the speedup offered by our method is particularly attractive. The case becomes even stronger in the context of stationary recursive equilibria, in which the idiosyncratic distribution Ω is time-invariant – C^j_l can be computed at the start of a simulation and used all throughout.

In our application, the aggregate state Z is degenerate due to reasons outlined at the end of Section 2.1, therefore it can be disregarded. After solving for k and r, we obtain C_l for labor by interpolating over grids of the idiosyncratic states (a, ε, ψ), and then use Eq. (15) to directly approximate n_t as a function of w. Notice that we no longer need to average over individual labor supply decisions with this approach.

3 Application

We compare the computational performance of the three TE solution methods across three dimensions: (1) the execution time of the aggregation procedure – in other words, the time it takes to find the sum of individual policy rules given all necessary inputs; (2) the execution time of the multidimensional mapping characterizing the TE, which needs to be computed at each step of the nonlinear solver; (3) the execution time of the nonlinear solver that approximates the set of equilibrium prices. We generate samples of these execution times and present corresponding summary statistics in Table 1. Notice that our method (Method 3) offers significant speedups across all of the performance dimensions. Furthermore, notice that Method 2 is impractically slow compared to the other methods in the given high-dimensional setting.

In addition to an execution speed comparison, we compare the accuracy with which our method estimates the aggregate labor supply schedule. In Fig. 1, we plot the percentage deviation of the labor supply schedule approximated using Methods 2 and 3[3] from that obtained using Method 1 over a large interval surrounding the steady state wage level. We find that our approach offers accuracy similar to Method 1, since the two labor supply schedules are practically identical.

	Method #1	Method #2	Method #3
Aggregation	0.032795726	42.00302928	3.98e-06
	(0.023769756)	(39.43680994)	(3.20e-06)
Temporary Eq. Mapping	0.06114025	43.34148712	3.99e-06
	(0.050119322)	(43.22936301)	(3.44e-06)
Nonlinear Solver	0.681620303	486.2593348	0.000336456
	(0.65654354)	(439.0028829)	(0.000322401)
Sample Size	1000	1000	1000

Table 1: Temporary equilibrium solution methods execution times. *Note:* Mean execution time in seconds, the with minimum execution time in parentheses.

[3]Note that Methods 2 and 3 must yield the same approximation.

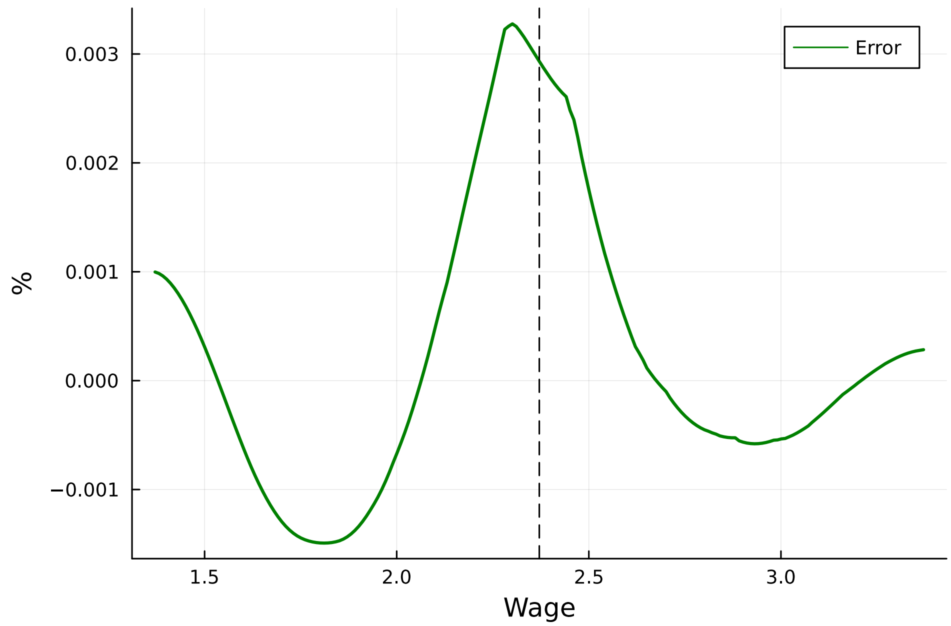

Figure 1: The percentage deviation of aggregate labor supply as a function of wage approximated using Method 3 from that obtained using Method 1. *Note:* The steady state wage level is represented by the dashed vertical line.

4 Conclusion

We develop a new method of approximating temporary equilibria in heterogeneous agent models by algebraically simplifying a conventional projection method. We compare the performance of existing methods of approximating temporary equilibria with our method by applying them to a model with heterogeneous boundedly rational agents presented in Evans et al. (2023). We find that our method offers a significant speedup without a notable drop in accuracy.

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